# 6 . Eigenvalue Oscillations in a Lake　Part 1

On the water surface of an inland lake or an inner bay, eigenvalue (proper) oscillations are sometimes induced by wind or incoming tsunamis, and such an eigenvalue　oscillation is referred to as “Seiche”

We assume a long wave approximation for such an oscillation, and we apply the following long wave equation:

$$\frac{\partial^2 \varsigma}{\partial t^2} = g\left\{\frac{\partial}{\partial x}\left(D\frac{\partial \varsigma}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial \varsigma}{\partial y}\right)\right\} \qquad (1)$$

We introduce a stationary solution as $\varsigma = Z(x,y)e^{-i\sigma t}$; we then have

$$-\sigma^2 Z = g\left\{\frac{\partial}{\partial x}\left(D\frac{\partial Z}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial Z}{\partial y}\right)\right\} \qquad (2)$$

There are two types of boundary conditions:

（a）Coastline boundary and（b）Open ocean boundary.

### （a） Coastline boundary

If we assume perfect reflection from the coast line, that is, if we assume that there is a vertical wall at the coast, there will be no velocity in the normal direction to the coastline. Hence,

$$\vec{v} \cdot \vec{n} = 0 \qquad (3)$$

where $\vec{n}$ is a unit vector parallel to the coastline.　The equation of motion is given as （$\vec{\ell}$ : normal direction of the coastline）：

$$\frac{\partial \vec{u}_\ell}{\partial t} = -g\frac{\partial \varsigma}{\partial \vec{l}} \qquad (4)$$

Here, $\vec{u}_\ell$ is zero; hence, (3) is equivalent to the following condition:

$$\frac{\partial \varsigma}{\partial \vec{n}} = 0 \qquad (5)$$

Equation (5) shows that the coastline behaves like a mirror for waves.

### （b）Open ocean boundary

We assume that any amount of water is supplied from the bay mouth as per the requirements from the inner bay.　Further, we assume that the open ocean is sufficiently deep and wide.

This assumption is equivalent to assuming that a nodal line, along which the sea level does not change, is present just outside the bay.

Thus,

$$\zeta = 0 \tag{6}$$

## 2. Circular lake with a constant depth

First, we consider water surface oscillation (Seiche) in a circular lake with a constant depth $(=D)$ and radius $a$. The equation of motion (2) can be re-written in a cylindrical coordinate system $(r,\theta,z)$ as

$$-\sigma^2 Z = c^2\left(\frac{\partial^2 Z}{\partial r^2}+\frac{1}{r}\frac{\partial Z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}\right) \tag{7}$$

where $c^2 = gD$. We place Z in the following manner

$$Z = W(r)\cos n\theta, \qquad n = 0,1,2,.... \tag{8}$$

Then, (7) becomes

$$W'' + \frac{1}{r}W' + \left(k^2 - \frac{n^2}{r^2}\right)W = 0 \qquad (\text{where } k^2 = \sigma^2 / c^2) \tag{9}$$

This is Bessel's differential equation, and it has the following general solution:

$$W(r) = C_1 J_n(kr) + C_2 N_n(kr) \tag{10}$$

Here $J_n, N_n$ are Bessel and Neumann functions. If we want to avoid a solution that the amplitude at the lake center is infinity, we should use $C_2 = 0$. Hence, the change in the lake surface is given by the following form:

$$\varsigma = J_n(kr)e^{-i\sigma t}\cos n\theta \tag{11}$$

If the radius of the lake is $a$, the coastline has $r = a$; hence,

$$\frac{dW}{dr} = 0 \qquad \text{at} \qquad r = a \tag{12}$$

On the other hand, the differential formula of Bessel function is

$$nJ_n'(x) = nJ_n(x)/x - J_{n+1}(x)$$

(See "Suugaku Koshiki", vol. 3, p159, Iwanami Press)

Using $J_n'(x) = 0$ at $x = ka$, we have

$$nJ_n(ka) = kaJ_{n+1}(ka) \tag{13}$$

By solving this equation numerically, we obtain k and σ $(= ck)$.

If we select only undirectional modes, that is, if we choose $n = 0$ solutions, (13) becomes

$$J_1(ka) = 0$$

$J_1(x)$ becomes zero at

$x = 0$ and $x_1 = 3.8317, x_2 = 7.0156, x_3 = 10.1347$........

Wave number $k$ assumes only the values of $k_i = x_i / a$, and the corresponding angular frequencies σ are

$$\sigma_i = ck_i = x_i \sqrt{gD} / a$$

Further, the period is given by

$$T_i = 2\pi a / (x_i \sqrt{gD}) \tag{14}$$

## 3. Seiche in a Rectangular Lake with Uneven Bottom

If we consider the Seiche in a rectangular lake and assume that the phenomenon is one-dimensional, the equation of motion is expressed as

$$\frac{\partial^2 \varsigma}{\partial t^2} = g \left\{ \frac{\partial}{\partial x} \left( D \frac{\partial \varsigma}{\partial x} \right) \right\} \tag{15}$$

In the case of a constant depth, we simply have a one-dimensional wave equation as follows:

$$\frac{\partial^2 \varsigma}{\partial t^2} = c^2 \frac{\partial^2 \varsigma}{\partial x^2} \quad (c^2 = gD) \tag{16}$$

We substitute $\varsigma = Z(x) e^{-i\sigma t}$, and then

$$Z'' + k^2 Z = 0 \tag{17}$$

where k is a wave number and $k = \sigma / c = 2\pi / L)$. $x = 0, a$, $Z' = 0$ should be satisfied at all sides of the lake; therefore,

$$Z = \cos \frac{n\pi x}{a} \tag{18}$$

By substituting this in (17), we find that $k$ takes only the following values:

$$k_n = \frac{n\pi}{a} \tag{19}$$

The corresponding wavelengths and periods are

$$L_n = 2a / n, \quad T_n = 2a / (n\sqrt{gD}) \tag{20}$$

If the depth is not always constant $D = D(x)$, we can apply Ritz's method. We substitute $\varsigma = Z(x) e^{-i\sigma t}$ into (15), and then we have

$$\frac{d}{dx}\left( D \frac{dZ}{dx} \right) + \frac{\sigma^2}{gD} Z = 0 \tag{21}$$

For an uneven bottom lake, we can apply Ritz's method; in other words, we obtain an approximate solution assuming that the boundary conditions are satisfied for all sides. We set such a function to have the following form:

$$Z = c_1 \cos\frac{\pi x}{a} + c_2 \cos\frac{2\pi x}{a} + c_3 \cos\frac{3\pi x}{a} + c_4 \cos\frac{4\pi x}{a} \tag{22}$$

According to Ritz's method, we calculate the value of the following integral

$$I = \int_0^a \left( DZ'^2 - \lambda \frac{\sigma^2}{gD} Z^2 \right) dx \tag{23}$$

and solve

$$\frac{\partial I}{\partial c_1} = 0, \frac{\partial I}{\partial c_2} = 0, \frac{\partial I}{\partial c_3} = 0, \frac{\partial I}{\partial c_4} = 0 \tag{24}$$

If we assume $c_1$ is a known number, then we obtain $c_2, ..., c_4$, and $\lambda$ which gives an approximate solution to the present problem.

[**Mathematical Note: Ritz's Method**

To numerically solve a second-order differential equation of the following type:

$$\left(p(x)y'\right)' + qy = r(x) \tag{25}$$

with a boundary condition

$$y(a) = y(b) = 0 \quad \text{or} \quad y'(a) = y'(b) = 0 \tag{26}$$

It is possible to obtain the approximate solution in the following manner:

(1) Determine a function set $\{\phi_i\}$ such that all its elements satisfy the boundary condition (26).

For example, $\{\phi_i\}= 1, \cos kx, \cos 2kx, \cos 3kx, ...$ (27)

(2) Assume the form of an approximate solution in the linear combination of the function set as

$$y = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + \cdots \tag{28}$$

(3) Calculate the following integral I

$$I = \int_a^b \left[ p(x)y'^2 - q(x)y^2 + 2r(x)y \right] dx \tag{29}$$

(4) Obtain $c_1, c_2, c_3, \cdots c_n$ by solving the following (linear) equations

$$\frac{\partial I}{\partial c_1} = 0, \frac{\partial I}{\partial c_2} = 0, \frac{\partial I}{\partial c_3} = 0, \cdots \frac{\partial I}{\partial c_n} = 0 \tag{30}$$

The theoretical basis for this method is derived from the principles applied in "Calculus of Variations" ("Henbun-Ho").